Topology

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Abstract

The notes are primarily based on *Topology* by James Munkres. The purpose is mainly helping me digest the materials.

1 Topological spaces and continuous functions

1.1 Topological spaces

Definition 1.1.1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$.
2. $\cup_{\alpha \in A} U_\alpha \in \mathcal{T}$ if $U_\alpha \in \mathcal{T}$ for all $\alpha \in A$.
3. $\cap_{i=1}^n U_i \in \mathcal{T}$ if if $U_i \in \mathcal{T}$ for all $1 \leq i \leq n$.

Example 1.1.2. See Example 1 on page 76.

Example 1.1.3. If X is any set, the **discrete topology** is the collection of all subsets of X.

Example 1.1.4. Let X be a set; let $\mathcal{T}_f := \{U \subset X : U^c \text{ is either finite or all of } X\}$ is called the **finite complement topology**.

1.2 Basis for topology

Definition 1.2.1. Let X be a set. A **basis** for a topology of X is a collection $\mathcal{B}$ of subsets of X such that

1. For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ such that $x \in B$.
2. If $x \in B_1 \cap B_2, B_1, B_2 \in \mathcal{B}$, then there is a basis element B_3 such that $x \in B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A subset $U \subset X$ is said to be **open** is for each $x \in U$, there is a basis element $B \subset \mathcal{B}$ such that $x \in B \subset U$. Clearly each $B \in \mathcal{B}$ is open.

Example 1.2.2. Let $X = \mathbb{R}^2$. Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane.

Example 1.2.3. Let $X = \mathbb{R}^2$. Let $\mathcal{B}$ be the collection of all rectangular regions (interiors of rectangles) in the plane.

Example 1.2.4. If X is any set, the collection $\mathcal{B}$ of all one-point subsets of X is a basis for the discrete topology in X.

Lemma 1.2.5. *If $\mathcal{T}$ be the topology generated by the basis $\mathcal{B}$, then $\mathcal{T} = \{\cup_{U \in \mathcal{I}} U : \mathcal{I} \subset \mathcal{B}$ is arbitrary$\}$.*

Lemma 1.2.6. *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{C} \subset \mathcal{T}$. If for each open set $U \in \mathcal{T}$ and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$, then $\mathcal{C}$ is a basis for $\mathcal{T}$.*

Proof. First we show that $\mathcal{C}$ is a basis. The first condition is satisfied by the fact that X is open. The second follows from observing that $C_1 \cap C_2$ is open if $C_1, C_2 \in \mathcal{C}$. Let $\mathcal{T}'$ denote the topology generated by $\mathcal{C}$. Since $O \in \mathcal{T}'$ is a union of elements of $\mathcal{C}$, it is open in $\mathcal{T}$ because $\mathcal{C} \subset \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary unions. This shows that $\mathcal{T}' \subset \mathcal{T}$. On the other hand, if $U \in \mathcal{T}$, then directly by definition U is open in the topology generated by $\mathcal{C}$. This establishes $\mathcal{T} = \mathcal{T}'$ and we are done. □

Lemma 1.2.7. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases for the toplogies $\mathcal{T}$ and $\mathcal{T}$. Then the following are equivalent:*

(a) $\mathcal{T} \subset \mathcal{T}'$, i.e., $\mathcal{T}'$ is finer than $\mathcal{T}$.

(b) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof. Clearly if $\mathcal{T} \subset \mathcal{T}'$, then each basis element $B \in \mathcal{B}$ is also an open in $\mathcal{T}'$. So by definition there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Conversely, because for each point in an given open set $O \in \mathcal{T}$, we can find a basis element $B \in \mathcal{T}$ such that $x \in B \subset O$. Then by assumption, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset O$. This implies that O is also open in $\mathcal{T}'$. Thus $\mathcal{T} \subset \mathcal{T}'$. □

Now we define three topologies on the real line all of which are of interest.

Definition 1.2.8. Let $\mathcal{B} := \{(a, b) : a, b \in \mathbb{R}$ and $a < b\}$. The topology generated by $\mathcal{B}$ is called the **standard topology** on the real line. Let $\mathcal{B}' := \{[a, b) : a, b \in \mathbb{R}$ and $a < b\}$. The topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$. Let $K := \{1/n : n \in \mathbb{Z}_+\}$ and $\mathcal{B}'' = \mathcal{B} \cup \{(a, b) \setminus K : a, b \in \mathbb{R}$ and $a < b\}$. The topology generated by $\mathcal{B}''$ is called the **K-topology** on $\mathbb{R}$.

Lemma 1.2.9. *The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.*

Proof. Let $\mathcal{T}, \mathcal{T}_\ell, \mathcal{T}_K$ denote these three topologies respectively. Clearly $\mathcal{T} \subset \mathcal{T}_\ell$. But any interval of the form $[a, b)$ is not open in $\mathcal{T}$ because there is no basis element $B \in \mathcal{B}$ such that $a \in B \subset [a, b)$. Similarly $\mathcal{T} \subset \mathcal{T}_K$. On the other hand, $(-1, 1) \setminus K$ is not open in the standard topology because 0 is not an interior point. □

Sometimes, we are interested in some more simpler structure of sets.

Definition 1.2.10. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Exercises

1. Let $\{\mathcal{T}_\alpha\}$ be a family of topology on X. Show that $\cap_\alpha \mathcal{T}_\alpha$ is a topology on X and $\cup_\alpha \mathcal{T}_\alpha$ is not necessarily a topology on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.
2. If $\mathcal{A}$ is a basis/subbasis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies that contain $\mathcal{A}$.
3. Show that $\mathcal{B} = \{(a, b) : a < b \text{ and } a, b \in \mathbb{Q}\}$ generates the standard topology on $\mathbb{R}$. Show that $\mathcal{C} = \{[a, b) : a < b \text{ and } a, b \in \mathbb{Q}\}$ generates a different topology than the lower limit topology on $\mathbb{R}$.
4. Problem 1,2,3,6,7 on Page 83.

1.3 Order topology

Definition 1.3.1. Let X be a set equipped with a total ordering $<$. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All intervals of the form $[a_0, b)$, there a_0 is the smallest element (if any) of X.
3. All intervals of the form $(a, b_0]$, there b_0 is the largest element (if any) of X.

The topology generated by the basis $\mathcal{B}$ is called the **order topology**.

Example 1.3.2. The standard topology on $\mathbb{R}$ is just the order topology derived from the usual order on $\mathbb{R}$.

Example 1.3.3. Consider $\mathbb{R} \times \mathbb{R}$ with the dictionary order. We denote the general element in $\mathbb{R} \in \mathbb{R}$ by $x \times y$. Clearly the set has neither the smallest nor the largest element so the order topology on $\mathbb{R} \times \mathbb{R}$ has as basis the collection

$$\{(a \times b, c \times d) : a < c\}$$

or

$$\{(a \times b, a \times d) : b < d\}$$

Example 1.3.4. $\mathbb{Z}_+$ form an ordered set with a smallest element. The order topology on $\mathbb{Z}_+$ is the discrete topology as $\{n\} = (n-1, n+1)$ and $\{1\} = [1, 2)$ are all basis elements.

Example 1.3.5. The set $X = \{1, 2\} \times \mathbb{Z}_+$ is another example of an ordered set with a smallest element. Denoting $1 \times n$ by a_n and $2 \times n$ by b_n, we can represent

$$X = \{a_1, a_2, ..., b_1, b_2, ...\}.$$

The order topology on X is **NOT** the discrete topology. Most one-point sets are open but $\{b_1\}$ is not open.

Definition 1.3.6. If X is an ordered set and $a \in X$. There are four subsets of X that are called **rays** determined by a: $(a, +\infty), [a, +\infty), (-\infty, a), (-\infty, a]$.

Lemma 1.3.7. *Open rays form a subbasis for the order topology.*

1.4 Product topology on $X \times Y$

Definition 1.4.1. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by the basis $\mathcal{B}$, where $\mathcal{B} = \{U \times V : U \subset X, V \subset Y \text{ are open}\}$.

We should note that $\mathcal{B}$ however is not in general a topology.

Theorem 1.4.2. *If $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for the topologies on X and Y respectively, then*

$$\mathcal{B} = \{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$$

is a basis for the product topology on $X \times Y$.

Proof. Clearly each element in $\mathcal{B}$ is open in the product topology. Let $O \subset X \times Y$ be open. For each $x \times y \in O$ there is a basis element $x \times y \in U \times V \subset O$. Since U is open in X and V is open in Y, we can find two basis elements B_X and B_Y of $\mathcal{B}_X$ and $\mathcal{B}_Y$ respectively such that $x \in B_X \subset U$ and $y \in B_Y \subset V$. Then clearly $B_X \times B_Y \in \mathcal{B}$ and $x \times y \in B_X \times B_Y \subset U \times V \subset O$. Hence $\mathcal{B}$ is a basis for the topology on $X \times Y$. $\square$

Example 1.4.3. Using this theorem, the standard topology on $\mathbb{R}^2$ is generated by the basis $\{(a, b) \times (c, d)\}$

Definition 1.4.4. The **projection mapping** $\pi_1 : X \times Y \to X$ is defined by $\pi(x, y) = x$ and projection π_2 is defined analogously.

Theorem 1.4.5. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. First note that $X \times Y = \pi_1^{-1}(X) = \pi_2^{-1}(Y) \in \mathcal{S}$. So $\mathcal{S}$ does satisfies the definition of a subbasis. Clearly each element in $\mathcal{S}$ is open in the product topology and this implies that the topology $\mathcal{T}'$ generated by $\mathcal{S}$ is coarser/weaker than than the product topology $\mathcal{T}$. On the other hand, the basis element of the product topology $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in \mathcal{T}'$. This implies $\mathcal{T} \subset \mathcal{T}'$ hence concludes the proof. $\square$

1.5 Subspace topology

Definition 1.5.1. Let Y be a subset of a topological space X with topology $\mathcal{T}$. The collection

$$\mathcal{T}_Y := \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on Y called the **subspace topology**.

Lemma 1.5.2. *If $\mathcal{B}$ is a basis for the topology of X then $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

Proof. First note that each element in $\mathcal{B}_Y$ is open in the subspace topology $\mathcal{T}_Y$. To show that $\mathcal{B}_Y$ is indeed a basis, suppose that $Y \cap U \in \mathcal{T}_Y$. For each $x \in Y \cap U$, because U is open in X, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$. So $x \in Y \cap B \subset Y \cap U$. This completes the proof. □

Lemma 1.5.3. *Let Y be a subspace of X. If U is open in Y and Y is open in X then U is open in X.*

Proof. If U is open in Y and Y is open in X, then $U = Y \cap O$ where both Y and O are open in X which implies U is open in X. □

Theorem 1.5.4. *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the subspace topology on $A \times B$.*

Proof. Let us denote $\mathcal{T}$ the subspace topology on $A \times B$ inherited from $X \times Y$ and denote $\mathcal{T}'$ the product topology on $A \times B$. Consider a basis element O_1 for $\mathcal{T}$. Then

$$O_1 = (A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$$

where U, V are open in X, Y respectively. The second equality shows that O_1 is also a basis element for $\mathcal{T}'$. Therefore $\mathcal{T}, \mathcal{T}'$ have the same basis elements which implies that $\mathcal{T} = \mathcal{T}'$. □

However, this equivalence is not true generally with order topology.

Example 1.5.5. Let $Y = [0, 1) \cup \{2\}$. In the subspace topology $\{2\}$ is open. However in the order topology on Y, $\{2\}$ is not open.

Example 1.5.6. Let $I = [0, 1]$. The dictionary order on $I \times I$ is just the restriction to $I \times I$ of the dictionary order on the plane $\mathbb{R} \times \mathbb{R}$. However, the dictionary order topology on $I \times I$ is not the same as the subspace topology on $I \times I$. The set $\{1/2\} \times (1/2, 1]$ is open in the subspace topology but not open in the dictionary topology on $I \times I$.

Given $Y \subset X$, we say Y is **convex** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lie in Y. Note that intervals and rays in X are convex in X.

Theorem 1.5.7. *Let X be an ordered set in the order topology; let $Y \subset X$ be convex. Then the order topology on Y is the same as subspace topology on Y.*

Exercises.

1. If Y is a subspace of X and $A \subset Y$, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.
2. If $\mathcal{T}, \mathcal{T}'$ are topologies on X such that $\mathcal{T}'$ is strictly finer than $\mathcal{T}$ and $Y \subset X$, then $\mathcal{T}'_Y$ is finer than $\mathcal{T}_Y$ but not necessarily strictly finer.
3. Projections are open map.
4. Let X, X' be single set in the topologies $\mathcal{T}, \mathcal{T}'$. Similarly for $Y, Y', \mathcal{S}, \mathcal{S}'$. If $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{S} \subset \mathcal{S}'$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$. The converse is also **TRUE**.
5. The countable collection

$$\{(a,b) \times (c,d) : a,b,c,d \in \mathbb{Q}\}$$

6. is a basis for $\mathbb{R}^2$.
7. A convex subset Y of an ordered set X is not necessarily an interval or a ray in X. The result is still **FALSE** even if X is a ray or an interval.
8. Problem 8,9,10 on page 92.

1.6 Closed set and limit point

Definition 1.6.1. A subset A of a topological space X is said to be **closed** if A^c is open.

Example 1.6.2. $[a,b], [a,+\infty), (-\infty,a]$ are closed.

Example 1.6.3. $A = \{(x,y) : x \geq 0, y \geq 0\}$ is closed because $A^c = \mathbb{R} \times (-\infty,0) \cup (-\infty,0) \times \mathbb{R}$ is open.

Example 1.6.4. In the finite complement topology on a set X, the closed sets consists of X and all finite subsets of X.

Example 1.6.5. In the discrete topology, every set is open. It follows that every set is closed as well.

Example 1.6.6. Consider $Y = [0,1] \cup (2,3)$ in the subspace topology. $[0,1]$ and $(2,3)$ are open and closed simultaneously.

Theorem 1.6.7. *Let X be a topological space. Then the following hold:*

1. *$X, \varnothing$ are closed.*
2. *Arbitrary intersections of closed sets are closed.*
3. *Finite unions of closed sets are closed.*

Proof. Taking complements in corresponding results for open sets will yield a proof. □

Definition 1.6.8. If Y is a subspace of X, we say a set $A \subset Y$ is **closed in** Y if $Y \setminus A$ is open in Y.

Theorem 1.6.9. *Let Y be a subspace of X. Then A is closed in Y if and only if $A = Y \cap C$ for some closed subset $C \subset X$.*

Proof. First suppose that A is closed in Y, i.e. $Y \setminus A$ is open in Y. That is $Y \setminus A = Y \cap O$ for some open set O in X. Clearly this implies $A = Y \cap O^c$ and O^c is closed in X as desired. Conversely, if $A = Y \cap C$ for some closed set $C \subset X$. Then $Y \setminus A = Y \cap (Y \cap C)^c = Y \cap C^c$ which implies that $Y \setminus A$ is open in Y. Hence A is closed in Y. □

Theorem 1.6.10. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Given a subset A of a topological space X, the **interior** of A is defined as the union of all open sets contained in A and the **closure** is defined as the intersection of all closed sets containing A.

Theorem 1.6.11. *Let Y be a subspace of X and $A \subset Y$. Then the closure of A in Y equals $\bar{A} \cap Y$.*

Proof. Let us denote the closure of A in Y by $\tilde{A}$. We observe that $\bar{A} \cap Y$ is closed in Y and contains A so by definition $\tilde{A} \subset \bar{A} \cap Y$. On the other hand, since $\tilde{A}$ is closed in Y, we have $\tilde{A} = Y \cap C$ for some closed set $C \subset X$. Moreover, C contains A, which implies $\bar{A} \subset C$. Thus $\bar{A} \cap Y \subset C \cap Y = \tilde{A}$. This concludes that $\tilde{A} = \bar{A} \cap Y$. □

Theorem 1.6.12. *Let A a subset of a topological space X.*

(a) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

Proof. (a) We shall prove the following equivalence:

$$x \notin \bar{A} \iff \text{There is an open set } U \text{ containing } x \text{ does not intersect } A.$$

Clearly if $x \notin \bar{A}$, the open set $X \setminus \bar{A}$ contains x and does not intersect A. On the other hand, suppose that there is an open set U containing x does not intersect A. Then note that $A \subset X \setminus U$ which is a closed set. By definition $\bar{A} \subset X \setminus U$. Thus it is obvious that $x \notin \bar{A}$.

(b) Suppose that $x \in \bar{A}$. Since every basis element B containing x is itself open, clearly B intersects A. Conversely, for each open set U containing x, we can find a basis element B such that $x \in B \subset U$. Then $B \cap A \neq \varnothing \Rightarrow U \cap A \neq \varnothing$ is immediate. □

Given a subset A of a topological space X, we say x is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself.

Theorem 1.6.13. $\bar{A} = A \cup A'$.

Proof. First we show that $\bar{A} \subset A \cup A'$. Pick $x \in \bar{A}$, then either $x \in A$ or $x \notin A$. In the first case, there is nothing to prove. If $x \notin A$, then by definition every neighborhood containing x intersects A, i.e. $x \in A'$. The reverse inclusion just follows from the previous theorem. □

Corollary 1.6.14. *A subset of a topological space is closed if and only if it contains all its limit points.*

Definition 1.6.15. A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1, U_2 of x_1, x_2 such that $U_1 \cap U_2 = \varnothing$.

Theorem 1.6.16. *Every finite point set in a Hausdorff space is closed.*

Proof. It suffices to show that every one-point set in a Hausdorff space is closed. We only need to show $\{a\}^c$ is open. Clearly for each $b \neq a$, there are two disjoint open sets U, V such that $a \in U$ and $b \in V$. Then V does not contain a implying that $V \subset \{a\}^c$. So $\{a\}^c$ is open. □

This condition is usually called T_1 **axiom**, which is weaker than the Hausdorff property. For example, the finite complement topology on the real line is T_1 but not Hausdorff.

Theorem 1.6.17. *Let X satisfies the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. The "only if" part is trivial. To prove sufficiency, suppose that there is a neighborhood U of x containing finitely many points of A. WLOG, we write $U \cap A = \{x_1, ..., x_n\}$. Note that the latter set is closed by T_1 axiom, so $U \setminus \{x_1, ..., x_n\}$ remains open and contains x. But U intersects A in no points other than x itself, i.e. x is not limit point of A. The concludes the contrapositive argument. □

Theorem 1.6.18. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Proof. Suppose $x_n \to x$ and $y \neq x$. Then there are two disjoint open sets U, V such that $x \in U$ and $y \in V$. In particular, there is a positive integer N_0 such that for all $n \geq N_0$, $x_n \in U$. But then for any positive integer N, there is always some $n \in N$ such that $x_n \notin V$ because $U \cap V = \varnothing$. This shows that $x_n \not\to y$. □

Theorem 1.6.19. *Every totally ordered set is a Hausdorff space in the order topology. The product of two Hausdorff space is Hausdorff. A subspace of a Hausdorff space is a Hausdorff space.*

Results from Exercises.

1. Let $\mathcal{C}$ satisfy that axioms of closed sets. The collection $\mathcal{T} = \{X \setminus C : C \in \mathcal{C}\}$ is a topology.
2. If A is closed in Y and Y is closed in X, then A is closed in X.
3. If A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.
4. If U is open in X and A is closed in X, then $U \setminus A$ is open and $A \setminus U$ is closed.

5. Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subset [a,b]$. If a,b are limit points of (a,b), or equivalently, if a does not have immediate successor and b does not have immediate predecessor, then the equality holds.
6. (a) If $A \subset B$, then $\bar{A} \subset \bar{B}$. (b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$. (c) $\cup_\alpha \bar{A}_\alpha \subset \overline{\cup_\alpha A_\alpha}$.
7. (a) $\overline{A \cap B} = \bar{A} \cap \bar{B}$. (b) $\overline{\cap_\alpha A_\alpha} = \cap_\alpha \bar{A}_\alpha$. (c) $\bar{A} \setminus \bar{B} \subset \overline{A \setminus B}$.
8. If $A \subset X$ and $B \subset Y$, then $\overline{A \times B} = \bar{A} \times \bar{B}$.
9. X is Hausdorff if and only if the diagonal $\triangle = \{x \times x : x \in X\}$ is closed in $X \times X$.
10. In the finite complement topology on $\mathbb{R}$ the sequence $x_n = 1/n$ converge to every point in $\mathbb{R}$.
11. T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.
12. Define the boundary $\partial A = \bar{A} \cap (\bar{A})^c$: (a) $A^\circ \cap \partial A = \varnothing$ and $\bar{A} = A^\circ \cup \partial A$. (b) $\partial A = \varnothing \Leftrightarrow A$ is both open and closed. (c) U is open iff $\partial U = \bar{U} \setminus U$. (d) If U is open, then $U \subset (\bar{U})^\circ$.

1.7 Continuous functions

Definition 1.7.1. Let X, Y be topological spaces. A function $f : X \to Y$ is **continuous** if for each open set $V \subset Y$, the preimage $f^{-1}(V) \subset X$ is open.

Remark 1.7.2. Clearly, if the topology on Y is given by a basis/subbasis, it suffices to check the preimage of basis/subbasis element under f is open in X.

Example 1.7.3. ε-δ definition of continuity in $\mathbb{R}$ coincides with this one.

Example 1.7.4. The identity map $f : \mathbb{R} \to \mathbb{R}_\ell$ is not continuous. However, the identity map $g : \mathbb{R}_\ell \to \mathbb{R}$ is continuous.

Theorem 1.7.5. *Let X, Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:*

(a) f is continuous.

(b) For every subset $A \subset X$, one has $f(\bar{A}) \subset \overline{f(A)}$.

(c) For every closed set $B \subset Y$, the set $f^{-1}(B)$ is closed in X.

(d) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Proof. We show in the following direction $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$ and $(a) \Leftrightarrow (d)$.

$(a) \Rightarrow (b)$. Suppose that f is continuous. Pick $x \in \bar{A}$. For each neighborhood V of $f(x)$, the preimage $f^{-1}(V)$ is open and contains x. By definition $f^{-1}(V)$ intersects A and this yields that $V \cap f(A)$ is nonempty. So $f(x) \in \overline{f(A)}$.

$(b) \Rightarrow (c)$. Let B be a closed set in Y. Clearly $f^{-1}(B) \subset \overline{f^{-1}(B)}$. We need to show the reverse direction. By assumption we have $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \bar{B} = B$. This implies that $\overline{f^{-1}(B)} \subset f^{-1}(f(\overline{f^{-1}(B)})) \subset f^{-1}(B)$. Hence $f^{-1}(B)$ is closed.

$(c) \Rightarrow (a)$. Let V be open in Y. Then V^c is closed in Y and $f^{-1}(V^c)$ is closed by assumption. Note that $f^{-1}(V) = (f^{-1}(V^c))^c$, so it is open in X. This justifies that f is continuous.

Next we show that $(a) \Leftrightarrow (d)$. Suppose that f is continuous. For each $x \in X$ and each neighborhood V of $f(x)$, the preimage $f^{-1}(V)$ is open. So there is a basis element U containing x such that $x \in U \subset f^{-1}(V)$. Then clearly $f(U) \subset V$. On the other hand suppose that (d) is true. It follows that for each $x \in f^{-1}(V)$, there is a neighborhood U_x of x such that $U_x \subset f^{-1}(V)$. Then we obtain that $f^{-1}(V) = \cup_{x\in f^{-1}(V)} U_x$. Because each U_x is open, $f^{-1}(V)$ is open as well. □

Definition 1.7.6. Let X, Y be topological spaces. If both $f : X \to Y$ and its inverse $f^{-1} : Y \to X$ are continuous, then f is called a **homeomorphism**.

Remark 1.7.7. Note that f is a homeomorphism is equivalent to say f is bijective and $f(U)$ is open if and only if U is open. Properties that are invariant under homeomorphisms are called **topological property**.

Example 1.7.8. $f(x) = 3x + 1$ and $f^{-1}(y) = 1/3(y - 1)$.

Example 1.7.9. $F(x) = \frac{x}{1-x^2}$ and $G(y) = \frac{2y}{1+\sqrt{1+4y^2}}$.

Example 1.7.10. Let S^1 denote the unit circle in $\mathbb{R}^2$. Let $f : [0, 1) \to S$ be such that $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Clearly f is bijective and continuous but f^{-1} is not continuous. For instance, $f([0, \frac{\pi}{4}))$ is not open in S^1.

Example 1.7.11. We may regard the function in the previous example as $f : [0, 1) \to \mathbb{R}^2$. Then f is continuous and injective but is not an imbedding.

Theorem 1.7.12 (Rules for constructing continuous functions)**.** *Let X, Y, Z be topological spaces.*

(a) (Constant function) If $f : X \to Y$ maps all of X into a single point $y_0 \in Y$, then f is continuous.

(b) (Inclusion) If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

(d) (Restricting the domain) If $f : X \to Y$ is continuous, and if A is a subspace of X then the restricted function $f\restriction_A : A \to Y$ is continuous.

(e) (Restricting or expanding the range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the range $f(X)$, then the function $g : X \to Z$ obtained by restricting the range is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range is continuous.

(f) (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f\restriction_{U_\alpha}$ is continuous for each α.

Proof. (a) Let U be an open set in Y. Then $f^{-1}(U) = X$ if $y_0 \in U$ and $f^{-1}(U) = \varnothing$ if $y_0 \notin U$. In both cases $f^{-1}(U)$ is open, whence f is continuous.

(b) Clearly $j^{-1}(U) = U \cap A$ which by the very definition is open in the subspace topology. This implies that j is continuous.

(c) Take an open set $U \subset Z$. Then $g^{-1}(U)$ is open in Y. It follows that $f^{-1}(g^{-1}(U))$ is open in X. Thus $g \circ f$ is open.

(d) Note $f\lceil_A = f \circ j$ where $j : A \to X$ is the inclusion function. By part (c), the restriction $f\lceil_A$ is continuous.

(e) Let $O \subset Z$ be open. We may find an open set U in Y such that $O = U \cap Z$. Since Z contains $f(X)$, $g^{-1}(O) = f^{-1}(O) = f^{-1}(O) \cap f^{-1}(Z) = f^{-1}(O)$ which is ope in X. Hence $g : X \to Z$ is continuous. Next let $j : Y \to Z$ be the inclusion function. We observe that $h = j \circ f$, so h is continuous.

(f) □

Theorem 1.7.13. *Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.*

The theorem also holds if A, B are open, which just follows from the local formulation of continuity.

Theorem 1.7.14 (Maps into products)**.** *Let $f : A \to X \times Y$ be given by the equations*

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous if and only if the functions

$$f_1 : A \to X \text{ and } f_2 : A \to Y$$

are continuous.

1.8 Product topology

Definition 1.8.1. Let J be an index set. Given a set X, we define a ***J*-tuple** of elements of X to be the function $\mathbf{x} : J \to X$. If $\alpha \in J$, we denote the value of $\mathbf{x}$ at α by $\mathbf{x}_\alpha$ rather than $\mathbf{x}(\alpha)$; we call it the αth **coordinate** of $\mathbf{x}$. We often denote the function $\mathbf{x}$ itself by the symbol $(x_\alpha)_{\alpha \in J}$. We denote the set of all J-tuples of elements of X by X^J.

Definition 1.8.2. Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \cup_\alpha A_\alpha$. The **cartesian product** of this indexed family, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions $\mathbf{x} : J \to X$ such that $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Clearly if all the sets $A_\alpha = X$, then $\prod_{\alpha\in J} A_\alpha = X^J$.

Definition 1.8.3. Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space $\prod_{\alpha\in J} X_\alpha$, the collection of all sets of the form $\prod_{\alpha\in J} U_\alpha$ where U_α is open in X_α for each $\alpha \in J$. The topology generated by this basis is called the **box topology**.

Let π_β be the projection onto the βth coordinate.

Definition 1.8.4. Let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ is open in } X_\beta\}$ and $\mathcal{S} = \cup_{\beta\in J}\mathcal{S}_\beta$. The topology generated by the subbasis $\mathcal{S}$ is called the **product topology**. In this topology $\prod_{\alpha\in J} X_\alpha$ is called a **product space**.

Theorem 1.8.5. *The box topology on $\prod X_\alpha$ has a basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ except for finitely many values of α.*

Immediately, for finite products $\prod_{\alpha=1}^n X_\alpha$, the two topologies are the same. In general the box topology is finer than the product topology.

Theorem 1.8.6. *Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. Then a basis for the box topology would be $\mathcal{B}_b = \{\prod B_\alpha : B_\alpha \in \mathcal{B}_\alpha\}$. A basis for the product topology would be $\mathcal{B}_p = \{\prod B_\alpha : B_\alpha \in \mathcal{B}_\alpha$ for finitely many $\alpha, B_\alpha = X_\alpha$ for the remaining indices$\}$.*

Theorem 1.8.7. *Let A_α be a subspace for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

Theorem 1.8.8. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

Theorem 1.8.9. *Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \in X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$.*

Theorem 1.8.10. *Let $f : A \to \prod_{\alpha\in J} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha\in J}$, where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.*

Example 1.8.11. The theorem will fail if we use the box topology. Let $X = \mathbb{R}^\omega$, the countable product of $\mathbb{R}$ with itself. Let $f : \mathbb{R} \to \mathbb{R}^\omega$ be $f(t) = (t, t, t, ...)$ where each of the coordinate function $f_n(t) = t$ is continuous on $\mathbb{R}$. But we claim that f is not continuous if $\mathbb{R}^\omega$ is given the box topology. Let $B = \prod_{n=1}^\infty(-1/n, 1/n) = (-1, 1) \times (-1/2, 1/2) \times \cdots$ and we show that $f^{-1}(B)$ is not open in $\mathbb{R}$. Clearly $0 \in f^{-1}(B)$ but there is no open neighborhood about the point 0 lying in $f^{-1}(B)$.

1.9 Metric topology

Definition 1.9.1. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ having the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 1.9.2. The collection of all ε-balls $B_d(x, \varepsilon)$ form a basis for a topology on X, called the **metric topology** induced by d.

Clearly a set U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

Definition 1.9.3. If X is a topological space, X is **metrizable** if there exists a metric d on X that induces the topology of X. A **metric space** is a metrizable space X together with a specific d that gives the topology of X.

Definition 1.9.4. Let (X, d) be a metric space. A subset $A \subset X$ is **bounded** if there is a number M such that $d(x, y) \leq M$ for all $x, y \in X$. If A is bounded and nonempty, the **diameter** of A is defined to be $\text{diam} A = \sup\{d(x, y) : x, y \in A\}$.

Note that boundedness is not a topological property for it depends on a particular metric.

Theorem 1.9.5. *Let (X, d) be a metric space. Define $\bar{d} : X \to X \to \mathbb{R}$ by the equation $\bar{d}(x, y) = \min\{d(x, y), 1\}$. Then $\bar{d}$ is a metric that induces the same topology as d.*

Definition 1.9.6. Given $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, we define the **norm** of $\mathbf{x}$ by the equation

$$||\mathbf{x}|| = \sqrt{x_1^2 + \cdots + x_n^2}$$

and we define the **euclidean metric** d on $\mathbb{R}^n$ by

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$$

We define the **square metric** ρ by

$$\rho(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$$

Lemma 1.9.7. *Let d, d' be two metrics on X; let $\mathcal{T}, \mathcal{T}'$ be topologies induced by d, d' respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$, each $\varepsilon > 0$, there exists $\delta > 0$ such that $B_d(x, \delta) \subset B_{d'}(x, \varepsilon)$.*

Theorem 1.9.8. *The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.*

Definition 1.9.9. Given an index set J and given points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\},$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. It is easy to check that $\bar{\rho}$ is a metric; it is called the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology**.

Theorem 1.9.10. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology these three topologies are all different if J is infinite.*

Theorem 1.9.11. *Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $\mathbf{x}, \mathbf{y}$ are two points of $\mathbb{R}^\omega$, define $D(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_i, y_i)/i\}$. Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.*

Theorem 1.9.12. *Let $f : X \to Y$ where (X, d_X) and (Y, d_Y) are metric spaces. Then f is continuous if and only if given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d_Y(f(x), f(y)) < \varepsilon \text{ whenever } d_X(x, y) < \delta.$$

Lemma 1.9.13. *Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x then $x \in \bar{A}$. The converse if true if X is metrizable.*

Theorem 1.9.14. *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n) \to f(x)$ in Y. The converse is true if X if metrizable.*

Lemma 1.9.15. *The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient operation is continuous function from $\mathbb{R} \times \mathbb{R} \setminus \{0\}$ into $\mathbb{R}$.*

Theorem 1.9.16. *If $f, g : X \to \mathbb{R}$ are continuous, then $f + g, f - g, f \cdot g$ are continuous. If $g \neq 0$ for all $x \in X$, then f/g is continuous.*

Definition 1.9.17. Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence f_n **converges uniformly** to $f : X \to Y$ if given $\varepsilon > 0$, there exists an integer $N > 0$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and for all $x \in X$.

Theorem 1.9.18 (Uniform limit theorem)**.** *Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $f_n \to f$ uniformly, then f is continuous.*

It is not hard to show that f_n converges uniformly to f if and only if f_n converges to f in the uniform metric $\bar{\rho}$.

Example 1.9.19. $\mathbb{R}^\omega$ in the box topology is not metrizable.

Example 1.9.20. An uncountable product of $\mathbb{R}$ with itself is not metrizable.

2 Connectedness and Compactness

2.1 Connected spaces

Definition 2.1.1. Let X be a topological space. A **separation** of X is a pair of nonempty disjoint open subsets $U, V \subset X$ such that $X = U \cup V$. The space X is **connected** if there is no separation of X.

Clearly, connectedness is a topological property. If X is connected, then any space homeomorphic to X is connected as well.

Remark 2.1.2. A moment of reflection shows that X is connected if and only if $X, \varnothing$ are the only subsets that are both open and closed.

To determine whether a subset $Y \subset X$ is connected or not, we need the following:

Lemma 2.1.3. *If Y is a subspace of X, a separation of Y is a pair of nonempty disjoint subsets $A, B \subset Y$ such that $Y = A \cup B$, and $\bar{A} \cap B = A \cap \bar{B} = \varnothing$. The space Y is connected if there is no separation of Y.*

Proof. Suppose that A, B is a separation of Y. By definition, A is both open and closed in Y. Then $A = \bar{A} \cap Y$; namely, $\bar{A} \cap B = \varnothing$. Similarly, we have $A \cap \bar{B} = \varnothing$. Conversely, suppose that A, B are given as in the assumption. Then $\bar{A} \cap B = \varnothing$ and $A \cup B = Y$ imply that $A = \bar{A} \cap Y$. Thus both A, B are closed in Y. But they are also open because $A = B^c$ and $B = A^c$. Thus A, B is a separation. □

Example 2.1.4. Let X be a two-point space in the indiscrete topology. Obviously X is connected.

Example 2.1.5. Let $Y = [-1, 0) \cup (0, 1]$. Then Y is not connected.

Example 2.1.6. $X = [-1, 1], A = [-1, 0], B = (0, 1]$. Note that A and B does **NOT** form a separation of X. Generally, there is no separation of the space $[-1, 1]$.

Example 2.1.7. The rational $\mathbb{Q}$ are not connected. Indeed, the only connected subspaces of $\mathbb{Q}$ are the one-point sets.

Example 2.1.8. $X = \{(x, y) : y = 0\} \cup \{(x, y) : x > 0 \text{ and } y = 1/x\}$. Clearly X is not connected.

Lemma 2.1.9. *If C, D form a separation of X, and if $Y \subset X$ is connected, then Y lies entirely within C or D.*

Proof. Consider two sets $C_Y = C \cap Y$ and $D_Y = D \cap Y$. Clearly C_Y, D_Y are open in Y and $C_Y \cup D_Y = Y$. If both of them are nonempty then C_Y, D_Y form a separation of Y which is a contradiction. Thus the result follows. □

Theorem 2.1.10. *If $\{A_i\}_{i \in I}$ is a collection of connected subspaces of X such that $\cap_{i \in I} A_i \neq \varnothing$, then $\cup_{i \in I} A_i$ is connected.*

Proof. Suppose C, D form a separation of $Y = \cup_{i\in I} A_i$. Let $p \in \cap_{i\in I} A_i$. Then either $p \in C$ or $p \in D$. Suppose $p \in C$, then $A_i \subset C$ for each $i \in I$ because A_i is connected for each i. This implies $D = \varnothing$ which yields a contradiction. □

Theorem 2.1.11. *Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.*

Proof. Suppose that C, D forms a separation of B. By Lemma 2.1.9, we may assume that $A \subset C$. Then $\bar{A} \subset \bar{C}$. But $\bar{C} \cap D = \varnothing$ so $B \cap D = \varnothing$ which contradicts to the fact that D is a nonempty subset of B. □

Theorem 2.1.12. *If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.*

Theorem 2.1.13. *A finite Cartesian product of connected spaces is connected.*

The counterpart for arbitrary products of connected space depends on which topology is used for the product.

Example 2.1.14.

Example 2.1.15.

2.2 Connected subspaces of the real line

Definition 2.2.1. A totally ordered set L having more than one element is a **linear continuum** if the following hold:

1. L has the least upper bound property.
2. If $x < y$, there exists z such that $x < z < y$.

Theorem 2.2.2. *If L is a linear continuum in the order topology, then L is connected and so are the intervals and rays in L.*

Corollary 2.2.3. *The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$*

Theorem 2.2.4 (Intermediate value theorem)**.** *Let $f : X \to Y$ be a continuous function where X is connected and Y is an ordered set in the order topology. If $a, b \in X$ and $r \in Y$ lying in between $f(a)$ and $f(b)$, then there exists a point c such that $f(c) = r$.*

Proof. Suppose to the contrary. Then $A = (-\infty, r) \cap f(X), B = f(X) \cap (r, +\infty)$ form a separation of $f(X)$. This contradicts to the fact that $f(X)$ is connected. □

2.3 Components and local connectedness

2.4 Compact spaces

Definition 2.4.1. A collection $\mathcal{A}$ of subsets of X is said to **cover** X, or to be a **covering** of X, if the union of the elements of $\mathcal{A}$ equals to X. An **open covering** of X is simply a covering of X whose elements are open subsets of X

Definition 2.4.2. A space X is **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that covers X (subcover).

Example 2.4.3. $X = \mathbb{R}$ is not compact. Consider $\mathcal{A} = \{(n, n+2) : n \in \mathbb{Z}\}$ which does not have any finite subcover.

Example 2.4.4. $X = \{0\} \cup \{1/n : n \in \mathbb{Z}_+\}$ is compact.

Example 2.4.5. Any space X containing only finitely many points is necessarily compact.

Example 2.4.6. $X = (0, 1]$ is not compact for $\mathcal{A} = \{(1/n, 1] : n \in \mathbb{Z}_+\}$ does not have any finite subcover.

Definition 2.4.7. If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to **cover** Y if the union *contains* Y.

Theorem 2.4.8. *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by opens set in X contains a finite subcollection covering Y.*

Proof. □

Theorem 2.4.9. *Every closed subspace of a compact space is compact.*

Proof. Let X be compact and $C \subset X$ be closed. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of C, $C \subset X$ is closed. Find a finite subcollection $\{U_i\}_{i=1}^n$ that covers X. Clearly $X \setminus C$ is open. Thus $\mathcal{A} = \mathcal{U} \cup \{X \setminus C\}$ forms an open covering of X. By compactness of X, we can find a finite subcollection $\{A_i\}_{i=1}^n$ which covers X. If $\{X \setminus C\} \in \{A_i\}_{i=1}^n$, then $\{A_i\}_{i=1}^n \setminus \{X \setminus C\} \subset \mathcal{U}$ is a finite subcollection which covers C. Otherwise the finite subcollection $\{A_i\}_{i=1}^n$ itself forms an open covering of C. In both cases, C is compact. □

Theorem 2.4.10. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space and $K \subset X$ be a compact subspace. We shall prove that $X \setminus K$ is open. Let $x \in X \setminus K$ and we will find an open neighborhood about x which is contained in $X \setminus K$. For each $y \in K$, we can use Hausdorff property to find two disjoint open sets U_y, V_y such that $x \in U_y, y \in U_y$ and $U_y \cap V_y = \varnothing$. Clearly $\{V_y\}_{y \in K}$ forms an open covering of K. Thus there is a finite subcollection $\{V_{y_i}\}_{i=1}^n$ which still covers K. Now let $V = \cup_{i=1}^n V_{y_i}$ and $U = \cap_{i=1}^n U_{y_i}$. It is immediate that $U \cap V = \varnothing$. But note that $K \subset V$ so $U \subset X \setminus K$ which is the desired open neighborhood about x contained in $X \setminus K$. □

In the preceding proof, we established a useful result:

Lemma 2.4.11. *If Y is a compact subspace of a Hausdorff space and $x_0 \notin Y$, then there exists disjoint open subsets U, V such that $x \in U$ and $Y \subset V$.*

Example 2.4.12. Once we showed that $[a, b]$ is a compact subset of $\mathbb{R}$, it directly follows that every closed subset of $[a, b]$ is compact. On the other hand, any interval of the form $(a, b), (a, b], [a, b)$ is not compact in $\mathbb{R}$ because they are not even closed in the Hausdorff space $\mathbb{R}$.

Example 2.4.13. One needs the Hausdorff property in the previous theorem. In fact, consider the finite complement topology on $\mathbb{R}$. Only finite subsets are closed whereas every subset is compact.

Theorem 2.4.14. *The image of a compact space under a continuous map is compact.*

Theorem 2.4.15. *If $f : X \to Y$ is a continuous bijection, X is compact and Y is Hausdorff, then f is a homeomorphism.*

Theorem 2.4.16 ($\star$)**.** *The product of finitely many compact spaces is compact.*

Proof. Let us prove that $X \times Y$ is compact if X, Y are compact. The general result simply follows from induction.

Suppose first that Y is compact and $x_0 \in X$, $N \subset X \times Y$ such that $x_0 \times Y \subset N$. We claim that

$$\textit{There is a neighborhood } W \textit{ of } x_0 \textit{ such that } W \times Y \subset N.$$

Since N is open, by definition, N equals to the union of a family of basis elements $U \times V$ where $U \subset X, V \subset Y$ is open. The family $\{U \times V\}$ also covers $x_0 \times Y$ which is compact, being homeomorphic to the compact space Y. Thus we can choose a finite subfamily $\{U_i \times V_i\}_{i=1}^n$ whose union contains $x_0 \times Y$. Denote $W = \cap_{i=1}^n U_i$ and we claim W is the desired open neighborhood of x_0. Let $x \times y \in W \times Y$. Note that $x_0 \times y \in U_i \times V_i$ for some $1 \le i \le n$, so $y \in V_i$ for some i. But $x \in U_i$ for every i, which implies $x \times y \in U_i \times V_i$ for some i. Hence $x \times y \in N$ because $U_i \times V_i \subset N$ for all i.

Now generally let $\mathcal{A} = \{A_i\}_{i \in I}$ be an open covering of $X \times Y$. Given $x \in X$, since $x \times Y$ is compact, we can find a finite subcover such that $x \times Y \subset \cup_{k=1}^n A_k =: N_x$. Now by previous argument, we see that there exists an open neighborhood W_x of x such that $W_x \times Y \subset N_x$. Now since $\{W_x : x \in X\}$ form an open covering of X, there exists an finite subcover $\{W_{x_j}\}_{j=1}^m$ which covers X. Each corresponding N_{x_j} is a finite subcollection of $\mathcal{A}$. Hence $\cup_{j=1}^m N_{x_j}$ is a desired open subcollection of $\mathcal{A}$ which covers $X \times Y$. This concludes that $X \times Y$ is compact. $\square$

Lemma 2.4.17 (The tube lemma)**.** *Let $X \times Y$ be a product space where Y is compact and $x \in X$ be any point. If $N \subset X \times Y$ is an open set such that $x \times Y \subset N$, then there is an open neighborhood W of x such that $W \times Y \subset N$.*

Definition 2.4.18. A collection $\mathcal{C}$ of subsets of X is said to have the **finite intersection property** (FIP) if for every finite subcollection $\{C_1, ..., C_n\} \subset \mathcal{C}$, the intersection $\cap_{i=1}^n C_i \neq \varnothing$.

Theorem 2.4.19. *Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the FIP, the intersection $\cap_{C \in \mathcal{C}} C$ is nonempty.*

Proof. This is just a set of linguistic equivalence.

Every collection $\mathcal{C}$ of closed sets in X having the FIP, the intersection $\cap_{C \in \mathcal{C}} C$ is nonempty.
$\Leftrightarrow$ If $\cap_{C \in \mathcal{C}} C = \varnothing$, then there is some finite subcollection $\{C_i\}_{i=1}^n \subset \mathcal{C}$ such that $\cap_{i=1}^n C_i = \varnothing$.
$\Leftrightarrow$ If $\cup_{C \in \mathcal{C}} C^c = X$, then there is some finite subcollection $\{C_i\}_{i=1}^n \subset \mathcal{C}$ such that $\cup_{i=1}^n C_i^c = X$.
$\Leftrightarrow$ X is compact.

The last equivalence follows from $\mathcal{C}$ is a collection of closed subsets of X if and only if $\mathcal{O} := \{X \setminus C : C \in \mathcal{C}\}$ is a collection of open subsets. □

2.5 Compact subspaces of the real line

Theorem 2.5.1 ($\star$)**.** *Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.*

Proof. Let $[a, b] \subset X$ be a closed interval and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover for $[a, b]$. We divide our proof in three steps:

Step 1. First we claim that given any $x < b$, there exists $y > x$ such that $[x, y]$ can be covered by at most two elements in $\mathcal{U}$. If x has an immediate successor, i.e. there is $y \in [a, b]$ such that $(x, y) = \varnothing$, then clearly $[x, y] = \{x, y\}$ can be covered by two elements in $\mathcal{U}$. If it is not the case, we know that there is U_i that contains x and U_i is open. So U_i contains an interval of the form $[x, y)$ for some $y \in [a, b]$.

Step 2. Let C denote the set of y such that $[a, y]$ can be covered by finitely many elements in $\mathcal{U}$. By step 1, C is not empty. Denote $c = \sup C$. We show that $c \in C$. Choose an element $U \in \mathcal{U}$ such that $c \in U$. Since U is open, it contains an interval of the form $(d, c]$. By definition, if $z < c$, then $z \in C$. Choose $d < z < c$, then $[a, z]$ can be covered by finitely many elements in $\mathcal{U}$. Note that $[z, c] \subset U$, so $[a, c]$ can be covered by finitely many elements in $\mathcal{U}$, i.e. $c \in C$.

Step 3. Finally we show that $c = b$. Suppose to the contrary that $c < b$. Apply step 1 with $x = c$, there is $y > c$ such that $[c, y]$ can be covered by finitely many elements in $\mathcal{U}$. But this implies $y \in C$ contradicts to $c = \sup C$. Hence $c = b$ and the proof is complete. □

Corollary 2.5.2. *Every closed interval in $\mathbb{R}$ is compact.*

Theorem 2.5.3. *A subspace $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded in the euclidean metric d or the square metric ρ.*

Proof. □

Theorem 2.5.4 (Extreme value theorem)**.** *Let $f : X \to Y$ be continuous where Y is an ordered set in the order topology. If X is compact, then there exists points c and d such in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

Definition 2.5.5. Let (X, d) be a metric space; let A be a nonempty subset of X. For each $x \in X$, we define the **distance from x to A** by

$$d(x, A) := \inf\{d(x, a) : a \in A\}$$

Clearly for fixed A, $d(\cdot, A)$ is a Lipschitz function of x as it satisfies

$$d(x, A) - d(y, A) \leq d(x, y).$$

Lemma 2.5.6 (The Lebesgue number lemma)**.** *Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, then there is a $\delta > 0$ such that for each subset $U \subset X$ with $\operatorname{diam} U < \delta$, there exists an element $A \in \mathcal{A}$ such that $U \subset A$.*

Proof. Let $\{A_i\}_{i=1}^n$ be a finite subcollection of $\mathcal{A}$ which covers X. Consider $\mathcal{C} := \{A_i^c : 1 \leq i \leq n\}$ and define $f(x) = \frac{1}{n}\sum_{i=1}^n d(x, C_i)$. We claim that $f(x) > 0$ for each $x \in X$. Find A_i that contains x, since A_i is open, we can find $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset A_i$. Then clearly $d(x, C_i) \geq \varepsilon$ and thus $f(x) \geq \varepsilon/n$. Now since f is continuous, by extreme value theorem, there is a minimum value $\delta := \min f(x)$. We shall show that δ is the Lebesgue number. Suppose $\operatorname{diam} B < \delta$ and $b \in B$. The definition of diam implies that $B \subset B_d(b, \delta)$. Moreover, the following inequality

$$\delta \leq f(b) \leq d(b, C_m),$$

where $d(b, C_m)$ the largest distance from b to $C_i, 1 \leq i \leq n$, implies that $B \subset B_d(b, \delta) \subset A_m$. This completes the proof. $\square$

Theorem 2.5.7 (Uniform continuity theorem)**.** *Let $f : X \to Y$ be a continuous function of the compact metric space (X, d_X) to the metric space (Y, d_X). Then f is uniformly continuous.*

Definition 2.5.8. A point $x \in X$ is said to be an **isolated point** of X if the one-point set $\{x\}$ is open in X.

Theorem 2.5.9. *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. First we observe that given any $x \in X$ and any nonempty open subset $U \subset X$, we can always find an open subset $V \subset U$ such that $x \notin \bar{V}$. Choose a different point $y \in U$. This can be done because: if $x \notin U$, then choose any $y \in U$ because U is nonempty; on the other hand if $x \in U$, then note x is not isolated, so there must exist some other point $y \in U$. Using Hausdorff property to find two disjoint open subsets W_1, W_2 such that $x \in W_1$ and $y \in W_2$. Since $\bar{W}_1, \bar{W}_2$ can be separated by disjoint open sets due to Hausdorff property, $V = W_2 \cap U$ is the desired open set. Now we claim that any $f : \mathbb{Z} \to X$ is not surjective. Let $x_n = f(n)$. We find an open set $V_1 \subset X$ such that $x_1 \notin \bar{V}_1$. Inductively, after finding V_{n-1}, we choose $V_n \subset V_{n-1}$ such that $x_n \notin \bar{V}_n$. Since X is compact, there is $x \in \cap_{n=1}^{\infty} \bar{V}_n$. But $x \neq x_n$ for all $n \in \mathbb{N}$ because $x_n \notin \bar{V}_n$ for all $n \in \mathbb{N}$. This completes the proof. $\square$

2.6 Limit point compactness

Definition 2.6.1. A space X is said to be **limit point compact** if every infinite subset of X has a limit point.

Theorem 2.6.2. *Compactness implies limit point compactness but not conversely.*

Theorem 2.6.3. *Let X be a metrizable space. Then the following are equivalent:*

1. *X is compact.*
2. *X is limit point compact.*
3. *X is sequentially compact.*

2.7 Local compactness

Definition 2.7.1. A space X is **locally compact at** x if there is a compact subspace C such that $x \in C$. If X is locally compact at each of its points, X is **locally compact.**

A general question is under what conditions is a space homeomorphic with a subspace of a compact Hausdorff space? The answer is given by the following theorem.

Theorem 2.7.2. *Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:*

1. *X is a subspace of Y.*
2. *The set $Y \setminus X$ consists of a single point.*
3. *Y is a compact Hausdorff space.*

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Remark 2.7.3. If X happens to be compact Hausdorff, then the space Y is obtained from X by adjoining a single isolated point. If X is not compact, then the point $Y \setminus$ is a limit point of X so that $\bar{X} = Y$.

Definition 2.7.4. If Y is a compact Hausdorff space and $X \subsetneq Y$ such that $\bar{X} = Y$, then Y is said to be a **compactification** of X. If $Y \setminus X$ equals a single point, then Y is called the **one-point compactification** of X.

Theorem 2.7.5. *Let X be a Hausdorff space. Then X is locally compact if and only if given $x \in X$, and given a neighborhood U of x, there is a neighborhood of V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.*

Proof. The sufficiency is clear. Now suppose that X is locally compact. Let $x \in X$ and U be a neighborhood of x. Take the one-point compactification Y of X. Consider $C = Y \setminus U$. Using Hausdorff property and compactness of C, we can find two open subsets V_1, V_2 in Y such that $x \in V_1$, $C \subset V_2$ and $V_1 \cap V_2 = \varnothing$. Clearly $\bar{V}$ is disjoint from C so $\bar{V} \subset U$ as desired. □

Corollary 2.7.6. *Let X be locally compact Hausdorff; let A be a closed or open subspace of X, then A is locally compact.*

Proof. If A is closed and $x \in A$, let U be a neighborhood of x such that $\bar{U} \subset X$ is compact. Then $U \cap A$ is ope neighborhood of x in A. Moreover, $\overline{U \cap A} \subset A$ is compact. If A is open, by previous theorem, we see that A is locally compact. □

Corollary 2.7.7. *A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.*

3 Countability and Separation Axioms

3.1 The countability axioms

Definition 3.1.1. A space X is said to have a **countable basis at** x if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least an element of $\mathcal{B}$. A space is said to satisfy the **first countability axiom**, or to be **first-countable** if it has a countable basis at each of its points.

Remark 3.1.2. Note that the collection $\mathcal{B}$ depends only on the point x instead of a neighborhood N of x.

Definition 3.1.3. X is said to be **second-countable** if it has a countable basis for its topology.

Remark 3.1.4. Clearly the first countable axiom implies the second.

Example 3.1.5. $\mathbb{R}, \mathbb{R}^n$ are both second-countable. In fact, the space $\mathbb{R}^\omega$ in the product topology is also second countable.

Example 3.1.6. In the uniform topology $\mathbb{R}^\omega$ is first countable but not second countable. Indeed, in a second countable space every discrete set/ set of isolated points is countable. But in $\mathbb{R}^\omega$, clearly the set of binary sequences is uncountable.

These two countability axioms are well behaved w.r.t. taking subspaces or countable products.

Theorem 3.1.7. *A subspace of a first-coutable space is first countable, and a countable product of first-countable spaces is first-countable. The same holds for second-countable spaces.*

Proof. Let X be a first countable space with the countable collection $\{B_n\}$ as in the definition and $A \subset X$ be a subspace. Then $\{B_n \cap A\}$ is the desired countable collection of neighborhoods in the subspace A.

Let $\prod_n X_n$ be a countable product of first-countable spaces. For each n, we let $\{B_{n,k}\}_{k=1}^\infty$ be the countable collection of neighborhoods in X_n. Clearly $\{\prod_n O_n : O_n \in \{B_{n,k}\}_{k=1}^\infty\}$ with all but finitely many $O_n = X_n$ is a desired collection. The proof for the second countability is similar. □

Definition 3.1.8. A subset A of a space X is **dense** if $\bar{A} = X$.

The following theorem characterizes some nice properties of second-countable spaces.

Theorem 3.1.9. *If X is second-countable, then (a) every open covering of X has a countable subcover; (b) there exists a countable dense subset in X.*

Proof. (a) Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open covering of X and let $\mathcal{B} = \{B_n\}_{n=1}^\infty$ be a countable basis for X. For each B_n we find an element U_n in $\mathcal{U}$ such that $B_n \subset U_n$. The collection $\{U_n\}$ is a countable subcollection of $\mathcal{U}$ that covers X.

(b) From each B_n we pick an arbitrary point b_n. The set $D = \{b_n : n \in \mathbb{N}\}$. We claim that $X = \bar{D}$. CLindelöflearly $\bar{D} \subset X$. The reverse inclusion follows from the fact that given any $x \in X$ and any basis element B_k of x contains the point $b_k \in D$; so $x \in \bar{D}$. □

Remark 3.1.10. A space satisfies the condition (a) is usually called a **Lindelöf space**. A space is said to be **separable** if it has a countable dense subset.

Example 3.1.11. The space $\mathbb{R}_\ell$ satisfies all countability axiom but the second. For each x, the collection $\{[x, x + 1/n)\}_{n\in\mathbb{N}^+}$ is countable basis at point x. So $\mathbb{R}_\ell$ is first countable.

To show that $\mathbb{R}_\ell$ is Lindelöf, we need more work. Indeed, it suffices to show that given any collection of basis elements which covers $\mathbb{R}_\ell$, there is a countable subcollection covering $\mathbb{R}_\ell$. Let $\mathcal{A} = \{[a_i, b_i)\}_{i\in I}$ be a collection of basis elements that covers $\mathbb{R}_\ell$. Consider $U = \cup_{i\in I}(a_i, b_i)$ and clearly $U^c = \{a_i\}_{i\in I}$. We claim that U^c is actually countable. Pick any $x \in U^c$ and we may write $x = a_j$ for some $j \in I$. Then choose any rational number $q_x \in (a_j, b_j)$. It is obvious that $(x, q_x) \subset (a_j, b_j)$. This implies if $x, y \in U^c$ with $x < y$, then $q_x < q_y$ for otherwise $x < y < q_y \leq q_x$ shows that $y \in (x, q_x) \subset U$ which is a contradiction. Hence we established an injection from U^c to a subset of $\mathbb{Q}$, so U^c is countable. Next for each element $a_i \in U^c$, we choose an element of $\mathcal{A}$ which contains a_i. In this manner, we find a countable subcollection $\mathcal{A}'$ of $\mathcal{A}$ that covers U^c. Now we may consider U as a subspace $\mathbb{R}$ in the standard topology. Since $\mathbb{R}$ is second countable, the subspace U is also second countable. In particular, the collection $\{(a_i, b_i)\}_{i\in I}$, by definition is an open cover of U. So there is a countable subcollection $\{(a_{i_k}, b_{i_k})\}_{k\in\mathbb{N}^+}$ which covers U. It follows that $\mathcal{A}'' = \{[a_{i_k}, b_{i_k})\}_{k\in\mathbb{N}^+}$ also covers U. Thus $\mathcal{A}' \cup \mathcal{A}''$ is a desired open subcover.

To see that $\mathbb{R}_\ell$ does not have a countable basis, let $\mathcal{B}$ be a basis for $\mathbb{R}_\ell$. For each $x \in \mathbb{R}_\ell$, we choose an element B_x such that $x \in B_x \subset [x, x + 1)$. Clearly $x \neq y$ implies $B_x \neq B_y$. Hence $\mathcal{B}$ contains uncountably many elements and $\mathbb{R}_\ell$ is not second-countable.

Example 3.1.12. The product of two Lindelöf spaces need not be Lindelöf. As we've seen, $\mathbb{R}_\ell$ is Lindelöf. But we now show the product $\mathbb{R}_\ell^2$ is not Lindelöf. Consider the closed subset

$$L = \{x \times (-x) : x \in \mathbb{R}_\ell\}.$$

We then cover L^c by the collection of basis elements $\mathcal{C} = \{[a, b) \times [-a, d) : a, b, d \in \mathbb{R}_\ell\}$. Clearly each element intersects L in at most one point $a \times (-a)$. Since L is uncountable, no countable subcollection $\mathcal{C}$ can cover $\mathbb{R}_\ell^2$.

Example 3.1.13. A subspace of s Lindelöf space need not be Lindelöf.

3.2 The separability axioms

Definition 3.2.1. Suppose that a space X is T_1, i.e. every one-point subset is closed. It is said to be **regular** (or a T_3 space) if given a pair of a point $x \in X$ and a closed subset $C \subset X$, there exists a pair of disjoint open subsets $U, V \subset X$ such that $x \in U$ and $C \subset V$. It is said to be a **normal** (or T_4) space if given a pair of two disjoint closed subsets $C_1, C_2 \subset X$ there exists a pair of open subsets containing C_1 and C_2, respectively.

Remark 3.2.2. The assumption that X is T_1 cannot be removed. Indeed, any two-point space with the trivial topology is both regular and normal but it is neither T_1 nor Hausdorff.

Lemma 3.2.3. *Let X be a T_1 space. Then,*

(a) X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A there is an open set V such that $A \subset V \subset \bar{V} \subset U$.

Proof. (a) Suppose first that X is regular. Let $x \in X$ and $U \subset X$ be a neighborhood of X. Clearly $x \notin U^c$ and U^c is a closed set. By regularity of X, there exists a pair of disjoint open sets $V_1, V_2 \subset X$ such that $x \in V_1$ and $U^c \subset V_2$. By replacing V_1 with $V_1 \cap U$ we may assume that $V_1 \subset U$. Then observe that for each $y \in U^c$, there is a neighborhood N of y such that $y \in N \subset V_2 \subset V_1^c$. So $y \notin \bar{V}_1$. It follows that that $U^c \subset \bar{V}_1^c$, i.e. $\bar{V}_1 \subset U$. Conversely, let $x \in X$ and closed set $C \subset X$ be given. Then the set C^c as an open neighborhood of x must contain an open set V such that $x \in V \subset \bar{V} \subset C^c$. Denote $V_1 = V, V_2 = \bar{V}^c$. Immediately, we have as desired $x \in V_1$, $C \subset V_2$ and $V_1 \cap V_2 = \varnothing$.

(b) Suppose that X is normal. Let U be an open set that contains a closed set A. Separating A, U^c, using normality of X, by two disjoint open sets $V_1, V_2 \subset X$. We may assume that $V_1 \subset U$. Then it is immediate that $\bar{V}_1 \subset U$ because $U^c \subset V_2 \subset \bar{V}_1^c$ and $V_2 \cap V_1 = \varnothing$. Conversely □

Theorem 3.2.4. (a) *A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff;*

(b) *A subspace of a regular space is regular; a product of regular spaces is regular;*

Proof. (a) Let X be a Hausdorff space and A be a subspace of X. If $a, b \in A$ and $a \neq b$, then we can find two disjoint open sets U, V in X such that $x \in U$ and $y \in V$. The sets $U \cap A, V \cap A$ then are two disjoints open sets in A that separate x, y.

Consider the product $\prod X_\alpha$ of Hausdorff spaces and distinct points $\mathbf{x}, \mathbf{y} \in \prod X_\alpha$. Clearly $x_\beta \neq y_\beta$ for some β. Use U_β, V_β to separate x_β, y_β. Then $\pi_\beta^{-1}(U_\beta), \pi_\beta^{-1}(V_\beta)$ are two disjoint open sets in $\prod X_\alpha$ that separate $\mathbf{x}$ and $\mathbf{y}$.

(b) Let X be regular and $A \subset X$ be a subspace. Suppose that $x \in A$ is an arbitrary point and $B \subset A$ is a closed subset in A that is disjoint from x. Clearly $B = \bar{B} \cap A$ and we can separate x and $\bar{B}$ and by open sets U, V in X, respectively. Then $U \cap A, V \cap A$ are disjoint open sets in A, and $x \subset U \cap A, B \subset V \cap A$ as desired.

Consider the product $\prod X_\alpha$ of regular spaces. Using previous lemma, we only need to show that given any $\mathbf{x} \in \prod X_\alpha$ and any open neighborhood U of $\mathbf{x}$ in $\prod X_\alpha$, there is an open neighborhood V of $\mathbf{x}$ such that $x \in V \subset \bar{V} \subset U$. Clearly we can find a open neighborhood of the form $\prod U_\alpha$ with all but finitely many $U_\alpha = X_\alpha$. For these finitely many α's, we use regularity of X_α, to choose an open neighborhood V_α of x_α such that $x_\alpha \in V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$. Set $V_\alpha = X_\alpha$ for the rest indices. Then clearly $\mathbf{x} \in \prod V_\alpha \subset \overline{\prod V_\alpha} = \prod \bar{V}_\alpha \subset \prod U_\alpha \subset U$, as desired. □

The following examples illustrate the fact that there are no analogous theorems for normal spaces.

Example 3.2.5. The space $\mathbb{R}_K$ is Hausdorff but not regular.

3.3 Normal spaces

In this section, we present several examples of normal spaces.

Theorem 3.3.1. *Every regular space with a countable basis is normal.*

Proof. Let A, B be two disjoint closed sets. For each point $a \in A$ we can find an open neighborhood U_a that is disjoint from B. The collection $\{U_a\}$ forms an open cover for A. Since the space X has a countable basis, it has a countable subcover $\{U_n\}$. Note that each U_n contains an open set whose closure does not intersect B. So, WLOG, we may assume that $\{U_n\}$ is a countable open covering of A and each $\bar{U}_n$ does not intersect B. Similarly we can choose an countable open cover $\{V_n\}$ of B where each $\bar{V}_n$ does not intersect A. Now for each n, consider $U'_n = U_n \setminus \left(\cup_{k=1}^n \bar{V}_k\right)$ and $V'_n = V_n \setminus \left(\cup_{k=1}^n \bar{U}_k\right)$. Observe that $A \subset \cup_n U'_n =: U'$ for each $x \in A$ we have $x \in U_n$ for some n but $x \notin \bar{V}_k$ for all k. Similarly $B \subset \cup_n V'_n =: V'$. Then we claim that $U' \cap V' = \varnothing$. If there is some $z \in U' \cap V'$, by definition there is some j, k such that $z \in U'_k \cap V'_j$. But this is impossible for any choices of j, k. Clearly U', V' are open. This completes the proof. □

Theorem 3.3.2. *Every metrizable space is normal.*

Proof. □

Theorem 3.3.3. *Every compact Hausdorff space is normal.*

Theorem 3.3.4. *Every well-ordered set is normal in the order topology.*

Remark 3.3.5. In fact more is true: every order topology is normal.

3.4 The Urysohn lemma

In this section, we prove the first "deep" result. In some sense, all previous work may be regarded as an introduction to topology vocabularies. This one, however, involves intricate originalities.

Theorem 3.4.1 (Urysohn lemma)**.** *Let X be a normal space; let A, B be two disjoint closed sets in X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous function $f : X \to [a, b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$.*

3.5 The Urysohn metrization theorem

Theorem 3.5.1. *Every regular space X with a countable basis is metrizable.*

3.6 The Tieze extension theorem

Theorem 3.6.1. *Let X be a normal space; let A be a closed subspace of X. Then*

(a) any continuous function from A into $[a, b]$ can be extended to a continuous function from X into $[a, b]$.

(b) Any continuous function from A into $\mathbb{R}$ can be extended to a continuous function from X into $\mathbb{R}$.

4 Complete Metric Spaces and Function Spaces

4.1 Complete metric spaces

Definition 4.1.1. Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ is said to be **Cauchy** if for every $\varepsilon > 0$, there is an positive integer N, such that $d_X(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. X is **complete** if every Cauchy sequence converges.

Lemma 4.1.2. *A metric space X is complete if every Cauchy sequence has a convergent subsequence.*

Theorem 4.1.3. *Euclidean space $\mathbb{R}^k$ is complete in either its usual metrics, the euclidean metric d, or the square metric ρ.*

Lemma 4.1.4. *Let $X = \prod X_\alpha$; let $\mathbf{x}_n$ be a sequence of points of X. Then $\mathbf{x}_n \to \mathbf{x}$ if and only if $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ for each α.*

Proof. The "only if" part follows from the continuity of each projection mapping. To show the converse, let U be an neighborhood of $\mathbf{x}$. Note that $U = \prod U_\alpha$ where all but finitely many U_α's are equal X_α's. For these finitely many open neighborhoods U_α, we can pick N large enough to make x_α's all lie within these neighborhoods. This implies for $\mathbf{x}_n \to \mathbf{x}$ as desired. □

Theorem 4.1.5. *There is a metric for the product space $\mathbb{R}^\omega$ relative to which $\mathbb{R}^\omega$ is complete.*

Proof. Recall that the product topology on $\mathbb{R}^\omega$ is metrizable via the metric $d = \sup\{\bar{d}(x_i, y_i)/i\}$. We claim that $(\mathbb{R}^\omega, d)$ is complete. Let $\mathbf{x}_n$ be a Cauchy sequence. By definition we have the following inequality

$$\bar{d}(\pi_i(\mathbf{x}_n), \pi_i(\mathbf{y}_n)) \leq i d(\mathbf{x}_n, \mathbf{y}_n).$$

This implies that for each $1 \leq i < \infty$, $(\pi_i(\mathbf{x}_n))_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathbb{R}, \bar{d})$, which indeed converges. By previous lemma, this shows that $\mathbf{x}_n$ converges as well. □

Remark 4.1.6. Recall that $\mathbb{R}^J$ when J is uncountable is not metrizable. So there is no analogous result for the product space $\mathbb{R}^J$. But under the uniform metric $\mathbb{R}^J$ is indeed complete, as the following result shows.

Recall the **uniform metric** on the metric space Y^J is defined by

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\}.$$

Theorem 4.1.7. *If the space Y is complete in the metric d, then the space Y^J is complete in the uniform metric $\bar{\rho}$.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence relative to the metric $\bar{\rho}$. Given any $j \in J$, we have

$$\bar{d}(f_n(j), f_m(j)) \leq \bar{\rho}(f_n, f_m).$$

This implies that $(f_n(j))_{n=1}^{\infty}$ is a Cauchy sequence relative to $\bar{d}$ for each $j \in J$; hence it converges, by completeness of Y, say, to y_j. Consider the function (or element of Y^J), defined by $f(j) = y_j$. We claim that $f_n \to f$ relative to $\bar{\rho}$. Given $\varepsilon > 0$, we can find $N \in \mathbb{N}^+$ such that,

$$\bar{d}(f_n(j), f_m(j)) < \varepsilon/2 \text{ for all } j \in J,$$

whenever $n, m \geq N$. Using continuity of $\bar{d}(\cdot, \cdot)$, and letting $m \to \infty$, we have

$$\bar{d}(f_n(j), f(j)) < \varepsilon/2 \text{ for all } j \in J.$$

Hence $\bar{\rho}(f_n, f) < \varepsilon$ for all $n \geq N$. This completes the proof. □

Theorem 4.1.8. *Let X be a topological space and let (Y, d) be a metric space. The set $C(X, Y)$ of all continuous functions from X to Y is closed in Y^X under the uniform metric. So is $B(X, Y)$ the set of all bounded functions. Hence, $C(X, Y), B(X, Y)$ are complete in their own right.*

Proof. The first part simply follows from the fact that uniform limit of continuous functions is continuous. Suppose that $f_n \to f$ uniformly and $(f_n)_{n=1}^{\infty}$ are bounded. Pick N so large that $\bar{d}(f_n(x), f(x)) < 1$ whenever $n \geq N$. We may assume that $\bar{d}(f_n(x), f_n(y)) < M$ for all $x, y \in X$. Then $\bar{d}(f(x), f(y)) \leq \bar{d}(f(x), f_n(x)) + \bar{d}(f_n(x), f_n(y)) + \bar{d}(f_n(y), f(y)) < M + 2$ which shows that f is a bounded function as well. □

On $B(X, Y)$, we can define another metric which is turned out to be equivalent to the uniform metric.

Definition 4.1.9. Suppose (Y, d) is a metric space. On $B(X, Y)$ we can define the **sup metric** via the equation

$$\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

We have a simple relation between the sup metric and uniform metric,

$$\bar{\rho}(f, g) := \min\{1, \rho(f, g)\}.$$

Thus it is clear that the sets of basis elements under these two metric are the same and they generate the same topology. In particular, if X is compact, then we usually use the sup metric on the space of all continuous functions in which case is a subspace of $B(X, Y)$.

We end this section by an interesting abstract theorem.

Theorem 4.1.10. *Let (X, d) be a metric space. There is an isometric imbedding of X into a complete metric space.*

4.2 A space-filling curve

4.3 Compactness in metric spaces

Definition 4.3.1. A metric space (X, d) is **totally bounded** if for every $\varepsilon > 0$, there is a finite covering of X by ε-balls.

Theorem 4.3.2. *Suppose that X is a metric space. Then X is compact if and only if X is complete and totally bounded.*

Proof. The "only if" direction is clear. We only prove the converse. Suppose that X is complete and totally bounded. We will show that X is sequentially compact. Let $(x_n)_{n=1}^{\infty}$ be an sequence of points in X. Since X is complete, we only need to find a Cauchy subsequence. First cover X by finitely many balls of radius 1. Denote B_1 the ball which contains infinitely many points of x_n's. Then shrinking the radius to $1/2$ and denote B_2 the ball that contains infinitely many x_n's lying in B_1. Continuing indefinitely in this fashion will yield a sequence of open balls $(B_n)_{n=1}^{\infty}$ of radius $1/n$. For each $k \in \mathbb{N}$, we choose $x_{n_k} \in B_k$ such that $n_k > n_{k-1}$. Then $(x_{n_k})_{k=1}^{\infty}$ is a Cauchy sequence as desired. □

Now we turn to prove compactness theorems in the space of all continuous functions. First we need to formulate a definition.

Definition 4.3.3. Let (Y, d) be a metric space. Let $\mathcal{F}$ be a subset of the function space $C(X, Y)$. If $x_0 \in X$, the set $\mathcal{F}$ is **equicontinuous at** x_0 if for all $\varepsilon > 0$, there is an neighborhood U of x_0 such that for all $x \in U$ and all $f \in \mathcal{F}$,

$$d(f(x), f(x_0)) < \varepsilon.$$

The set $\mathcal{F}$ is **equicontinuous** if it is equicontinuous at every point $x \in X$.

Lemma 4.3.4. *If $\mathcal{F} \subset C(X, Y)$ where is totally bounded under the uniform metric, then $\mathcal{F}$ is equicontinuous under d.*

Lemma 4.3.5. *Suppose that X is compact space and (Y, d) is a compact metric space. If $\mathcal{F} \subset C(X, Y)$ is equicontinuous under d, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to d.*

Definition 4.3.6. Let (Y, d) be a metric space. A subset $\mathcal{F} \subset C(X, Y)$ is **pointwise bounded** under d if for each $x \in X$ the subset $\mathcal{F}_a = \{f(a) : f \in \mathcal{F}\} \subset Y$ is bounded under d.

Theorem 4.3.7 (Ascoli's theorem, classical version)**.** *Let X be a compact space; let $(\mathbb{R}^n, d)$ denote the Euclidean space in either the square metric or the Euclidean metric; given $C(X, \mathbb{R}^n)$ the corresponding uniform topology. Then a subspace $\mathcal{F} \subset C(X, \mathbb{R}^n)$ has a compact closure if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded under d.*

Corollary 4.3.8. *Under the same assumption. A subspace $\mathcal{F} \subset C(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded under the sup metric ρ and equicontinuous under d.*

4.4 Pointwise and compact convergence

Definition 4.4.1. Given a point $x \in X$ and an open set U of the space Y, let

$$S(x, U) = \{f \in Y^X : f(x) \in U\}.$$

The sets $S(x, U)$ are a subbasis for topology on Y^X, which is called the **topology of pointwise convergence**.

Remark 4.4.2. Note that the topology of pointwise convergence is nothing new. It is the same as the product topology on Y^X because each subbasis element $S(x, U) = \pi_x^{-1}(U)$.

Next we can reformulate the result of convergence in product spaces using the functional notation.

Theorem 4.4.3. *A sequence $f_n \to f$ in the topology of pointwise convergence if and only if for each $x \in X$, the sequence $f_n(x) \to f(x)$ in Y.*

Example 4.4.4. Define $f_n(x) = x^n$ on $[0, 1]$. Clearly $f_n \to f$ pointwise where $f(x) = 1$ if $x = 1$ and $f(x) = 0$ otherwise. This shows that $C([0, 1], \mathbb{R})$ is not closed in the topology of pointwise convergence.

There are other topologies on the space of all functions from X to Y in which the subspace $C(X, Y)$ is closed.

Definition 4.4.5. The **topology of compact convergence** is a topology generated by the basis element $B_C(f, \varepsilon) := \{g \in Y^X : \sup_{x \in C} d(f(x), g(x)) < \varepsilon\}$ given a function $f \in Y^X$, a compact set $C \subset X$ and $\varepsilon > 0$.

We can check that the collection $\mathcal{B} := \{B_C(f, \varepsilon) : f \in Y^X, C\text{compact}, \varepsilon > 0\}$ does form a basis. Clearly for any function $f : X \to Y$ we can find some basis element $B_C(f, \varepsilon)$ that contains f. If $h \in B_C(f, \varepsilon_1) \cap B_D(g, \varepsilon_2)$, we can take $\delta = \min\{\varepsilon_1, \varepsilon_2\} - \max\{\sup\{d(f(x), h(x)) : x \in C\}, \sup\{d(g(x), h(x)) : x \in D\}\}$. Then $B_{C \cup D}(h, \delta) \subset B_C(f, \varepsilon_1) \cap B_D(g, \varepsilon_2)$ as desired.

Theorem 4.4.6. *A sequence $f_n \to f$ in the topology of compact convergence if and only if for each compact subspace $C \subset X$, the sequence $f_n\restriction_C$ converges uniformly to $f\restriction_C$.*

Proof. Note that $f_n \to f$ in the topology of compact convergence if and only if for every basis element $B_C(f, \varepsilon)$ containing f; that is given compact subset C, and $\varepsilon > 0$, there is a positive integer $N \in \mathbb{N}$ such that $f_n \in B_C(f, \varepsilon)$ for all $n \geq N$. This is equivalent to say given a compact set C and $\varepsilon > 0$, there is a positive integer N such that $\sup_{x \in C} d(f_n(x), f(x)) < \varepsilon$ for all $n \geq N$, i.e. $f_n\restriction_C$ converges uniformly to $f\restriction_C$. □

Definition 4.4.7. A space X is **compactly generated** if it satisfies the following condition: a set A is open in X if $A \cap C$ is open in C for each compact subspace C of X.

Lemma 4.4.8. *If X is locally compact, or if X satisfies the first countability axiom, then X is compactly generated.*

Proof. First let us suppose that X is locally compact. Let A be a subset having the property that $A \cap C$ is open in C for each compact subspace C of X. We show that A is open in X. Pick $x \in A$. There is an open neighborhood U of x such that $\bar{U}$ is compact. In particular $A \cap \bar{U}$ is open in $\bar{U}$ by assumption. That is for each point $p \in A \cap \bar{U}$, there is an basis element $O_p \cap \bar{U} \subset A \cap \bar{U}$. This implies that for each point

$p \in A \cap U$, $p \in O_p \cap U \subset A \cap U$. So $A \cap U$ is open in U hence is open in X. Then for each $x \in A$, there is a neighborhood $A \cap U \subset U$. So A is open.

Next suppose that X satisfies the first countability axiom. Let B be a set having the property that $B \cap C$ is closed for each compact set C. We shall show that B is closed. It suffices to show that if $x \in \bar{B}$, then $x \in B$. Since X is first-countable, it has a countable basis at x. So there is a sequence $x_n \in B$ such that $x_n \to x$. Note that $C = \{x\} \cup \{x_n\}$ is compact and by assumption $B \cap C$ is closed. Since $B \cap C$ contains infinitely many x_n's, so it must contain x as well. Hence $x \in B$ and B is closed. □

Lemma 4.4.9. *Suppose that X is compactly generated and $f \in Y^X$. If $f\restriction_C$ is continuous for each compact set $C \subset X$, then $f : X \to Y$ is continuous.*

Proof. Let $V \subset Y$ be open. We show that $f^{-1}(V)$ is open in X. Note that for each compact set C, the set

$$f^{-1}(V) \cap C = (f\restriction_C)^{-1}(V),$$

is open by continuity of $f\restriction_C$. Since this holds for all compact set C and X is compactly generated, the set $f^{-1}(V)$ is open as well. Hence f is continuous. □

Theorem 4.4.10. *Let X be a compactly generated space and let (Y, d) be a metric space. Then $C(X, Y)$ is a closed in Y^X in the topology of compact convergence.*

Proof. Let f be a limit point of $C(X, Y)$. Due to previous lemma, we only need to show that $f\restriction_C$ is continuous for each compact set $C \subset X$. This is just true because on each C, f is a uniform limit of a sequence of continuous functions hence is continuous. □

Corollary 4.4.11. *Let X be a compactly generated space and let (Y, d) be a metric space. If $f_n \to f$ uniformly on every compact set, then f is continuous.*

4.5 Ascoli's theorem

Theorem 4.5.1. *Let X be a space and let (Y, d) be a metric space. Given $C(X, Y)$ the topology of compact convergence; let $\mathcal{F}$ be a subset of $C(X, Y)$.*

(a) If $\mathcal{F}$ is equicontinuous under d and the set $\mathcal{F}_a = \{f(a) : f \in \mathcal{F}\}$ has a compact closure for each $a \in X$, then $\mathcal{F}$ is contained in a compact subspace of $C(X, Y)$.

(b) The converse holds if X is locally compact Hausdorff.

5 Baire Spaces and Dimension Theory

5.1 Baire spaces

Definition 5.1.1. A space X is said to be a **Baire space** if the following condition holds: given any countable collection $\{A_n\}_{n=1}^{\infty}$ of closed sets of X, each of which has empty interior in X, their union $\cup_{n=1}^{\infty} A_n$ also has empty interior in X.

Example 5.1.2. The space $\mathbb{Q}$ is not a Baire space.

Theorem 5.1.3. *X is a Baire space if and only if given any countable collection of $\{U_n\}_{n=1}^{\infty}$ of open dense subsets of X, their intersection is also dense in X.*

Theorem 5.1.4 (Baire category theorem)**.** *If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.*

Proof. (X is compact Hausdorff). Let $\{U_n\}_{n=1}^{\infty}$ be a collection of closed sets that have empty interior and let $U = \cup_{n=1}^{\infty} U_n$. We shall show that U has empty interior as well. That is, given any open set O, we need to find $x \in O$ that does not lie in any U_n at all. Let us write $V_n = U_n^c$ and so each V_n is open. Since U_1 has empty interior, we can find $y \in O \cap V_1$ which is open. In particular, since X is compact Hausdorff, we can find a neighborhood W_1 of y such that $\bar{W}_1 \subset O \cap V_1$. Inductively, after finding W_{n-1}, we can construct an open set W_n such that $\bar{W}_n \subset W_{n-1} \cap V_n$. Since X is compact Hausdorff, the intersection $\cap_n \bar{W}_n$ is nonempty and $x \in \cap_n \bar{W}_n$ has the property that $x \in V_n = U_n^c$ for all n. Thus the theorem follows.

(X is a complete metric space). We follow a similar construction but we require that $\operatorname{diam} W_n < 1/n$ additionally. This completes the proof if we use the following lemma. □

Lemma 5.1.5. *If $C_1 \supset C_2 \supset \cdots$ where each C_n is a closed subset in a complete metric space X and $\operatorname{diam} C_n \to 0$, then $\cap_n C_n \neq \varnothing$.*

Lemma 5.1.6. *Any open subspace of a Baire space is Baire.*

Theorem 5.1.7. *Let X be a Baire space and let (Y, d) be a metric space. If $f_n \to f$ pointwsire, then the set of points at where f is continuous is dense in X.*